Solving Quadratic Equations

- How do you know you have a quadratic equation?
  - The **highest power** (exponent) in the equation is 2 (a square term).
  - You have an **equation**: in other words, there is an equal sign.
  - Example: $x^2 – x + 1 = x + 4$

- Goal:
  - Find values of x that make the equation true.
  - For a quadratic equation (with highest power 2) you can have two different solutions, two equal solutions, or no solution.

- Look for the Special Case First
  - **Sometimes you get lucky.** If the quadratic equation is in the form (square term) = constant, then solve directly.
    - Example: $(x – 1)^2 = 4$
    - Here you have a square term $(x – 1)^2$ set equal to a constant (4).
    - The inverse operation for "squaring" is "square root."
    - Take the square root of each side of the equation. **Be careful**: when you take a square root **you get two terms – one positive and one negative**. We show that by including a plus-or-minus sign.

    $\sqrt{(x-1)^2} = \pm\sqrt{4}$ → $(x – 1) = \pm 2$

    - That gives <u>two</u> equations: $x – 1 = +2$ and $x – 1 = –1$.
    - Then, solve for x. That gives <u>two</u> solutions: $x = 3$ and $x = –1$.
  - **Sometimes you can make yourself lucky.** If the quadratic equation can be easily <u>put</u> in the form (square term) = constant, then solve directly.
    - Example: $x^2 – 2x + 1 = 4$
    - If you notice that $x^2 – 2x + 1 = (x – 1)^2$, then you have the same problem as in the previous example.

- The More General Case
  - If you're not so lucky (as described above), follow an alternative approach.
    - Goal: rearrange the quadratic equation so that it is in the form (first expression) * (second expression) = 0.
      - Why does that help solve the equation? Because you have two expressions multiplied together and equal to 0.
      - How can that happen? Only when either the first expression or the second expression or both are each equal to 0.
  - Steps:
    - Use simple algebra to get all variables and numbers on one side of the equal sign. 0 will then be left on the other side of the equal sign.
    - Factor the resulting equation. See the separate Factoring handout for details.
    - Set each factored expression = 0 and solve for x.
    - Example: $x^2 - x + 1 = x + 4$.
      - First, make the right-hand side (the simpler side) = 0. Subtract x and subtract 4 from each side of the equal sign.

$$\begin{array}{rcr} x^2 - x + 1 & = & x + 4 \\ - x - 4 & & - x - 4 \\ \hline x^2 - 2x - 3 & = & 0 \end{array}$$

      - Factor the left-hand-side:
        $x^2 - 2x - 3 = (x - 3)(x + 1) = 0$
      - Set each factored expression = 0:
        $x - 3 = 0$ and $x + 1 = 0 \rightarrow x = 3$ and $x = -1$
      - Note: this is the same answer as we got for the first example above.

- The Most General Case
  - If you're really not so lucky and can't figure out how to factor the quadratic equation, then use the **quadratic formula** (always works).
    - Quadratic equation in standard form: $ax^2 + bx + c = 0$.
    - Quadratic formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$; this gives two possible solutions for x.
      - Example: $x^2 - 3x + 1 = 0$ – this can't be easily factored. In this case a = 1, b = –3, and c = 1.
      - Insert values into the quadratic formula:
        $$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(1)}}{2(1)} = \frac{3 \pm \sqrt{9-4}}{2} = \frac{3 \pm \sqrt{5}}{2}$$
      - This gives <u>two solutions</u>: $x = \frac{3+\sqrt{5}}{2}$ and $x = \frac{3-\sqrt{5}}{2}$
    - What happens if the part under the radical sign ($b^2 - 4ac$) is zero? Then you will get only **one solution** for x.
      - Example: $x^2 - 4x + 4 = 0$. In this case a = 1, b = –4, and c = 4.
      - Insert values into the quadratic formula:
        $$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(4)}}{2(1)} = \frac{4 \pm \sqrt{16-16}}{2} = \frac{4 \pm \sqrt{0}}{2} = \frac{4}{2} = 2$$
      - This gives <u>one solution</u>: x = 2.
    - What happens if the part under the radical sign ($b^2 - 4ac$) is negative? Then you will get only **no real solutions** for x.
      - Example: $x^2 - 2x + 4 = 0$. In this case a = 1, b = –2, and c = 4.
      - Insert values into the quadratic formula:
        $$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(4)}}{2(1)} = \frac{4 \pm \sqrt{4-16}}{2} = \frac{4 \pm \sqrt{-12}}{2}$$
      - This gives <u>no real solutions</u>, because the square root of a negative number is not real.
        - Note: if you learn about "complex numbers", which allow the square root of –1, then you will always get two solutions, even in the last example above. However, both solutions will be complex numbers, not real numbers.