Polynomial Analysis

Definition: a **root** (also called **zero** or **solution**) of a function f(x) is a number ***a*** such that f(a) = 0.
Definition: if a is a root of a polynomial function f(x), then (x – a) is a **factor** of f(x)
Definition: if a is a root of a polynomial function f(x), then (a, 0) is an **x-intercept** of f(x)

Given: a polynomial of degree n: $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$

Step 1:
- Determine the highest power (n); this is the **degree** of the polynomial
- Based on the fundamental theorem of algebra you know that there will be <u>n roots</u>

Step 2:
- Determine the **leading coefficient** (a_n)
- Based on leading coefficient test you know the <u>left-hand and right-hand behaviors</u>:
  - If n is odd and $a_n > 0$, then the graph decreases to the left and increases to the right (N shape).
  - If n is odd and $a_n < 0$, then the graph increases to the left and decreases to the right (backwards N shape).
  - If n is even and $a_n > 0$, then the graph increases to the left and increases to the right (W shape).
  - If n is even and $a_n < 0$, then the graph decreases to the left and increases to the right (M shape).

Step 3:
- Determine the <u>candidate</u> **rational roots** of the polynomial
- The rational root theorem tells you that any candidate rational root will be in the form of a fraction $\frac{p}{q}$, where p is any factor of a_0 and q is any factor of a_n (including <u>negatives</u>).
  - Use <u>long division</u> or <u>synthetic division</u> to verify if a candidate root is an actual root.
    - For simple candidate roots (e.g., 1, –1, etc.) you can also substitute into f(x) and determine if f(x) = 0.
  - If none of the candidate roots are actual roots, then there are no rational roots.

Step 4:
- **Factor** the polynomial as much as possible
  - A polynomial function with real coefficients can always be factored into either <u>linear</u> (e.g., 2x – 1) or <u>quadratic</u> (e.g., $3x^2 + x + 1$) factors with real coefficients. The quadratic factors, if they can't be further factored into linear factors, have no real roots.
  - If you're given or know an <u>existing factor</u> (**linear**), use **synthetic division** to simplify the original polynomial. If you're given an existing factor (**quadratic**), use **long division** to simplify the original polynomial.
  - For any quadratic factor, use the <u>quadratic formula</u> to find <u>real or imaginary roots</u>.

Step 5:
- **Graph** the polynomial
  - Plot any know real roots
    - If there is a multiple root [e.g., $(x + 1)^2$], then the graph <u>crosses</u> at that root if the power is odd and <u>touches</u> if the power is even
  - Find the y-intercept by finding f(0)

Step 6 (Other Special Tests):

- **Descartes Rule of Signs**
  - Count the sign changes between each term. Then subtract 2 repeatedly until <= 0. The number of **positive roots** is one of the <u>non-zero</u> numbers.
  - Find f(–x) and then count the sign changes between each term. Then subtract 2 repeatedly until <= 0. The number of **negative roots** is one of the <u>non-zero</u> numbers.
- **Intermediate Value Theorem**
  - If during synthetic division with a <u>positive</u> candidate root you get a result with all non-negative values, then all real roots are less than the candidate root.
  - If during synthetic division with a <u>negative</u> candidate root you get a result where the <u>signs alternate</u>, then all real roots are greater than the candidate root.
- **Complex Zeros in Pairs**
  - If you find or are given one complex root ***a* + *bi***, then its conjugate ***a* – *bi*** is also a root.