Rational Functions - Asymptotes

Vertical Asymptotes

- Vertical asymptotes occur when the bottom (denominator) is 0, because then you are dividing by 0 (not allowed!).
- Therefore, set the bottom = 0 and solve. You'll need to factor the numerator (top) and denominator (bottom) of the fraction. The vertical asymptotes are of the form **x = a**, where **a** is a real number.
- Example: $\frac{2x-1}{x^2-1} = \frac{2x-1}{(x+1)(x-1)} \Rightarrow set\ (x+1)(x-1)=0,\ giving\ x=-1,1$

  $\Rightarrow vertical\ asymptotes\ are\ x=-1\ and\ x=1$
- Note: if identical factors appear in the numerator (top) and denominator (bottom) of the fraction, the corresponding solution represents a "hole", not an asymptote.
  - Example: $\frac{x-1}{x^2-1} = \frac{x-1}{(x+1)(x-1)} = \left\{\frac{1}{x+1},\ x \neq 1\right\} \Rightarrow asymptote\ at\ x=-1\ and\ hole\ at\ x=1$
  - When graphed, a hole appears as a missing point (a "hole", shown as an open circle at the missing point).

Horizontal Asymptotes

- **Top-heavy – no horizontal asymptote**

  Example: $\frac{x^3+1}{x+1} \Rightarrow no\ horizontal\ asymptote$

- **Bottom-heavy – horizontal asymptote: y = 0**

  Example: $\frac{x+1}{x^2+1} \Rightarrow horizontal\ asymptote:\ y=0$

- **Neutral – horizontal asymptote: y = p/q** (coefficients of highest powers in top and bottom)

  Example: $\frac{2x^2+1}{3x^2+1} \Rightarrow horizontal\ asymptote:\ y=\frac{2}{3}$

Slant (Linear) Asymptotes

- Slant asymptotes occur when the bottom (denominator) is one degree less than the top (numerator).
- Divide the top by the bottom to get the equation of slant asymptote, which is linear.
- Example: $\frac{2x^2-x-2}{x+1} = 2x-3+\frac{1}{x+1} \Rightarrow the\ fractional\ term \to 0\ as\ x \to \infty\ (or\ -\infty)$

  $\Rightarrow the\ slant\ asymptote\ is\ y=2x-3$

Graphing Using Asymptotes

- The horizontal and vertical asymptotes form the **framework** for generating the graph.
- Graph the horizontal and vertical asymptotes using dashed lines before graphing the function.
- Find any x-intercepts by setting y = 0. Find any y-intercepts by setting x = 0.
- On each side of each vertical asymptote the function approaches either $+\infty$ or $-\infty$.
- At the right hand side and left hand side of the graph, the function approaches the horizontal asymptote.