Equations of a Plane

General

There are many ways to find the equation of a plane for a set of given information. I'll discuss 3:

1. Given a single point and a vector perpendicular to the plane.
2. Given a single point and two vectors parallel to the plane.
3. Given three points.

Equations of a Plane Given A Point and a Vector Perpendicular to the Plane

- The key in this case is that any vector parallel to the plane is perpendicular to the given perpendicular vector **N**. The vector **N** is also called the "**normal vector**."
- Example: given a point A = (2, 4, 1) in a plane and a vector **N** = < – 21, 12, 17 > perpendicular to the plane, identify another arbitrary point P (x, y, z) in the plane.
- Because the given vector **N** is perpendicular to the plane, it is also perpendicular to any vector parallel to the plane. In particular, such a vector **AP** is parallel to the plane because both A and P are points actually in the plane. **AP** = < x – 2, y – 4, z – 1 >.
- Because the vectors **N** and **AP** are perpendicular, their dot product = 0.
  - **N** · **AP** = < – 21, 12, 17 > · < x – 2, y – 4, z – 1 > = 0.
  - Therefore, – 21 (x – 2) + 12 (y – 4) + 17 (z – 1) = 0.
  - This is the **Cartesian equation** of the plane. You can multiply out the terms to put it in the form –21x + 12y +17z = 23.
- There are an **infinite number of parametric equations** for this plane. You can construct one such very easily. You will need two parameters (say, s and t) to do so.
  - Start with the equation – 21 (x – 2) + 12 (y – 4) + 17 (z – 1) = 0.
  - Set y – 4 = s and z – 1 = t. Substituting above gives – 21 (x – 2) + 12s + 17t = 0.
  - Rearranging these three equations gives one set of **parametric equations**:

    $x = 2 + \frac{12s}{21} + \frac{17t}{21}$

    y = 4 + s

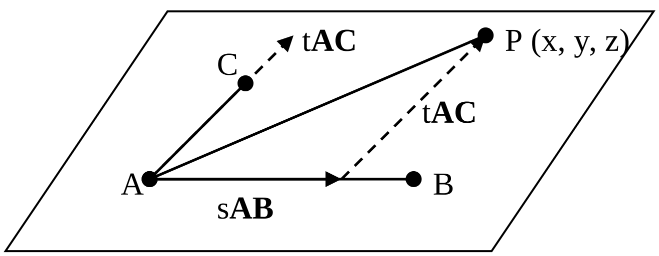
    z = 1 + t
- There are an **infinite number of vector equations** for this plane. You can construct one such very easily. You will use the parametric equation above to identify 3 vectors.
  - The first vector is from the origin to the given point. That vector is **a** = < 2, 4, 1 >.
  - The second vector is composed of the coefficients of s; call it **b** = < $\frac{12}{21}$, 1, 0 >.
  - The third vector is composed of the coefficients of t; call it **c** = < $\frac{17}{21}$, 0, 1 >.
  - The second and third vectors are called "direction vectors."
  - Call the vector describing the plane **p**. Then **p** = **a** + s**b** + t**c**. This is the vector equation of the plane.

Equations of a Plane Given A Point and Two Vectors Parallel to the Plane

- The key in this case is that the parallel vectors, assuming they are distinct, have a cross product that is perpendicular to the plane. You can then use the approach from the previous case to determine the equations.
- Example: given a point A = (2, 4, 1) in a plane and two vectors **u** = < –3, –1, –3 > and **v** = < –1, –6, 3 >, both of which are parallel to the plane, determine the cross product.
- Using the cross product approach used in a separate memo, **u** x **v** = < –21, 12, 17 >. Look familiar?
- Using the previous approach given a point and a normal vector, the **Cartesian equation** of the plane is – 21 (x – 2) + 12 (y – 4) + 17 (z – 1) = 0. A parametric equation and a vector equation are shown in the previous section.

Equations of a Plane Given Three Points in the Plane

- If you know three points (A, B, C) in a plane, you can find parametric and vector equations of the plane.
- Here's an example with A = (2, 4, 1), B = (–1, 3, –2), C = (1, –2, 4).
- First, identify the vectors **AC** and **AB**. **AB** = < –3, –1, –3 > and **AC** = < –1, –6, 3 >.
- Then identify the vector **AP** from A to an arbitrary point P(x, y, z) on the plane. Then, **AP** = < x – 2, y – 4, z – 1 >. See the picture below.



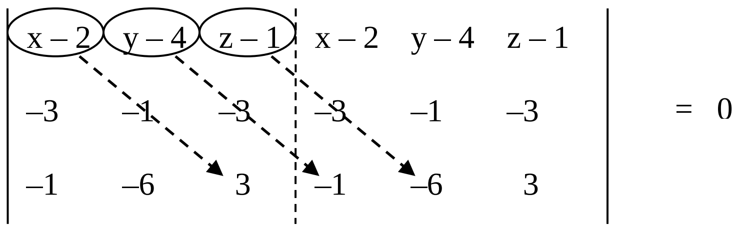
- **Claim**: the vector **AP** = s·**AB** + t·**AC**. This is the vector equation of the plane.
  - You can see this in the picture above. I've drawn the vectors s·**AB** and t·**AC** and showed that adding them head to tail gives the resultant **AP**.
- In this example < x – 2, y – 4, z – 1 > = s < –3, –1, –3 > + t < –1, –6, 3 >.
- Match up the components to get the parametric equation of the plane:
  - x – 2 = –3s – t → *x = 2 –3s – t*
  - y – 4 = –s – 6t → *y = 4 – s – 6t*
  - z – 1 = –3s + 3t → *z = 1 – 3s + 3t*
  - Note that the constants in the italicized equations are just the coordinates of point A. The coefficients of s are just the components of the vector **AB**. The coefficients of t are just the components of the vector **AC**.
- If you solve each of the three parametric equations for s and t in terms of x, y, and z, you can determine the Cartesian equation of the plane. There are many ways to solve three equations with three unknowns. However, a simple approach uses the **determinant of a matrix,** which we set = 0.

$$\begin{vmatrix} x-2 & y-4 & z-1 \\ -3 & -1 & -3 \\ -1 & -6 & 3 \end{vmatrix} = 0$$

You can compute this determinant using the same approach as we did when computing the cross product of two vectors (separate memo).

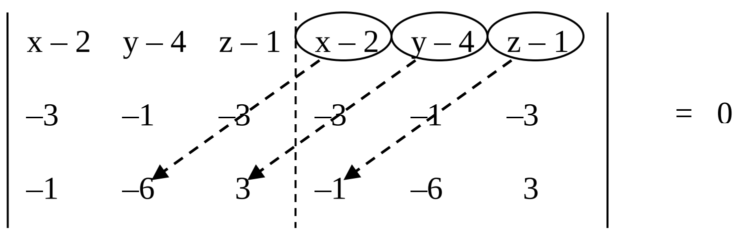
First, set up the first row of the matrix. It is just (x – a), (y – b), and (z – c), where (a, b, c) is the coordinate of the point A (2, 4, 1). The second row is the components of **AB** <–3, –1, –3 >. The third row is the components of **AC** < –1, –6, 3 >. These last two are again <u>direction vectors</u>.

- Now, generate the determinant in the same fashion as was done for a cross product. Start with multiplying the diagonal elements from left to right to get the positive terms.



Get the positive terms by multiplying on the diagonals.
That gives –3(x – 2) + 3(y – 4) + 18 (z – 1).

- Then, multiply the diagonal elements from right to left to get the negative elements.



Get the negative terms by multiplying on the diagonals. Negate each term.
That gives –18(x – 2) + 9(y – 4) – (z – 1).

- Add the positive and negative terms and set = 0 to get:

$$-21(x-2) + 12(y-4) + 17(z-1) = 0.$$

This is the Cartesian equation of the plane. You can multiply out the terms to put it in the form –21x + 12y +17z = 23.